

Quantum phase transition of polaritonic excitations in a multi-excitation coupled array

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We analyze the quantum phase transition-like behavior in the lowest energy state of a two-site coupled atom-cavity system, where each cavity contains one atom but the total excitation number is not limited to two. Utilizing the variance of the total excitation number to distinguish the insulator and superfluid states, and the variance of the atomic excitation number to identify the polaritonic characteristics of these states, we find that the total excitation number plays a significant role in the lowest-energy-state phase transitions. In both the small hopping regime and the small atom-field interaction regime, we identify an interesting coexisting phase involving characteristics of both photonic superfluid and atomic insulator. For small hopping, we find that the signature of the photonic superfluid state becomes more pronounced with the increase in total excitation number, and that the boundaries of the various phases shift with respect to the case of $N = 2$. In the limit of small atom-field interaction, the polaritonic superfluid region becomes broader as the total excitation number increases. We demonstrate that the variance of the total excitation number in a single site has a linear dependence on the total excitation number in the large-detuning limit.

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I. INTRODUCTION

Strongly correlated phenomena in controllable quantum many-body systems have attracted great attention in optical lattices [1, 2] and Josephson-junction arrays [3]. One of the simplest and most important models describing these light-matter interactions is the Jaynes-Cummings-Hubbard (JCH) model [4–6], which describes a coupled array of cavities each containing a two-level system. Similar to the Bose-Hubbard (BH) model [7, 8] used for cold atoms in an optical lattice, the JCH model exhibits the photon blockade effect [9, 10] and the Mott-insulator-to-superfluid quantum phase transition [11].

Coupled cavity-QED arrays can operate at high temperatures and allow for individual site addressing. These features, together with progress in realizing the strong light-matter coupling regime in both atomic and solid-state systems [12, 13], are attracting more and more attention to the quantum phase transitions in coupled-cavity arrays captured by the JCH model. These quantum phase transitions are due to the transfer of excitations from polaritonic states to photonic states, where polaritons are superpositions of photons and excitations of the atoms or atom-like structures, rather than purely bosonic or purely fermionic excitations. Most previous studies related to quantum phase transitions in coupled-cavity arrays have focused on the large site number and large atom number limits [14–17], which are analogous to the purely bosonic BH model and can be analytically solved within the mean-field approximation.

Recent research shows evidence that novel quantum phase transition-like behavior may appear in finite sys-

tems involving a very few interacting sites and a small number of two-level systems [18–22], where the transition behavior becomes dependent on the number of sites and two-level systems. Greentree *et al.* [4] showed that the Mott-insulator to superfluid quantum phase transition could appear in a mesoscopic two-dimensional coupled array. Hartmann *et al.* [5] used a four-level atom to simulate the effective on-site potential and the Mott-insulator to superfluid phase transition. Angelakis *et al.* [6] considered the simulation of an XY spin model based on the Mott regime in a linear array of cavities, each containing a two-level atom and a photon. Irish *et al.* [18] demonstrated phase transitions of polaritonic excitations in a two-site, two-excitation coupled array. In a very recent ion-trap experiment [23], Toyoda *et al.* reported the simulation of the quantum transition of polaritonic excitations in a JCH model using two trapped ions and phonons within the two-excitation Hilbert subspace. Previous works on the two-site coupled array model were limited to the two-excitation Hilbert subspace without considering the situation with higher excitation numbers in the photonic states. It is known that the number of photons plays an important role in the coefficient of on-site repulsion for small finite systems [4], but the influence of extra photons on the quantum phase transition in the lowest energy state of the coupled atom-cavity system is still unclear.

In this work, we investigate quantum phase transition-like behavior in the lowest energy state (within the N -excitation Hilbert subspace) of a two-site coupled atom-cavity system, where each cavity contains one atom but the total excitation number N is not limited to two. (Note that the descriptions “insulator” and “superfluid” used in this paper represent the localized and delocalized states in the small finite system we consider. Such an investigation can also be generalized to larger arrays, in

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which the localization-delocalization transitions studied here approach genuine quantum phase transitions.) By not restricting the system to the two-excitation subspace, the requirement for cooling the cavity field in experimental realizations is loosened. Our work is also applicable to ion-trap setups [23], where the photons are replaced by phonons. Section II establishes the model and its Hamiltonian. In Sec. III we carry out an extensive analysis, both analytically and numerically, of the lowest-energy-state properties for total excitation number $N = 4$. The limits of small hopping and of small atom-field interaction are both considered in detail. In Sec. IV the analysis is generalized to the case of higher excitation numbers. We briefly conclude in Sec. V.

II. SYSTEM HAMILTONIAN

We consider the system consisting of two sites, each supporting a field mode and containing a single atom. Photons are able to hop between these two field modes. Under the rotating-wave approximation, our system is governed by the following Hamiltonian ($\hbar = 1$):

$$H = \sum_{j=1,2} [w_c a_j^\dagger a_j + w_a |e_j\rangle\langle e_j| + \lambda(a_j^\dagger |g_j\rangle\langle e_j| + a_j |e_j\rangle\langle g_j|)] + h(a_1^\dagger a_2 + a_1 a_2^\dagger), \quad (1)$$

where a_j^\dagger and a_j are the creation and annihilation operators of the j th field mode with frequency w_c . $|e_j\rangle$ and $|g_j\rangle$ represent the excited and ground states of the j th atom with frequency w_a . λ is the atom-field coupling strength and h is the strength of the hopping between the two cavity fields. Note that the excitation number of the total system is conserved since the excitation number operator $\hat{N} = \sum_{j=1,2} (|e_j\rangle\langle e_j| + a_j^\dagger a_j)$ commutes with the Hamiltonian H .

III. TOTAL EXCITATION NUMBER $N = 4$

A. Small hopping

When there is no photon hopping between the two sites, the eigenstates in each site are given by the polaritonic states [18] :

$$|0_j\rangle = |g_j\rangle|0_j\rangle, \quad (2)$$

$$|n_j^-\rangle = \sin\left(\frac{\theta_n}{2}\right)|e_j\rangle|(n-1)_j\rangle - \cos\left(\frac{\theta_n}{2}\right)|g_j\rangle|n_j\rangle, \quad (3)$$

$$|n_j^+\rangle = \cos\left(\frac{\theta_n}{2}\right)|e_j\rangle|(n-1)_j\rangle + \sin\left(\frac{\theta_n}{2}\right)|g_j\rangle|n_j\rangle, \quad (4)$$

where $j = 1, 2$. $|n_j\rangle$ ($n = 1, 2, 3, \dots$) represents the Fock state of the j th field mode, and $\tan(\theta_n) = 2\lambda\sqrt{n}/\Delta$,

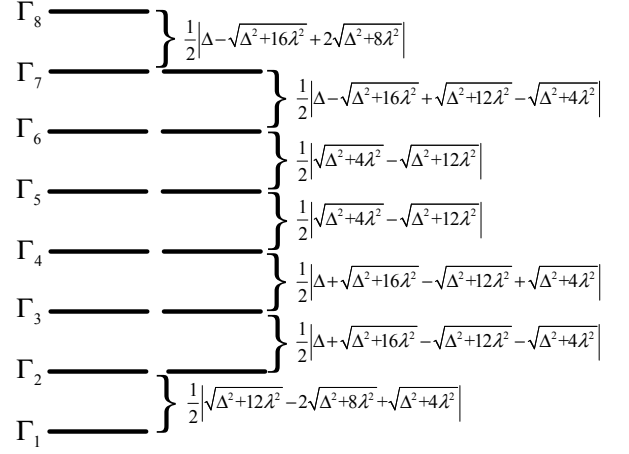


FIG. 1: Energy-level difference between two nearest neighbor subspaces when there is no photon hopping and $N = 4$.

where the detuning $\Delta = w_a - w_c$. The corresponding energies for these eigenstates are:

$$E_j^0 = 0, \quad (5)$$

$$E_j^{n-} = nw_c + \frac{\Delta}{2} - \frac{1}{2}\sqrt{\Delta^2 + 4n\lambda^2}, \quad (6)$$

$$E_j^{n+} = nw_c + \frac{\Delta}{2} + \frac{1}{2}\sqrt{\Delta^2 + 4n\lambda^2}. \quad (7)$$

Unlike the previous study in Refs. [18, 23] where the analysis is restricted to the Hilbert subspace with only two excitations, our analysis here focuses on the insulator-superfluid quantum phase transition in the multi-excitation Hilbert space, i.e., the total excitation number N can be larger than two. For simplicity, we consider the total excitation number to be an even number N , for which the lowest energy state is nondegenerate. In the following, we begin the analysis of quantum phase transition in the lowest energy state of our atom-cavity system for the case $N = 4$ and later generalize it to $N > 4$.

When $h = 0$ and $N = 4$, the eigenstates of the system are, arranged in order of increasing energy,

$$\Gamma_1 = \{|2_1^-\rangle \otimes |2_2^-\rangle\}, \quad (8)$$

$$\Gamma_2 = \{|1_1^-\rangle \otimes |3_2^-\rangle, |3_1^-\rangle \otimes |1_2^-\rangle\}, \quad (9)$$

$$\Gamma_3 = \{|0_1\rangle \otimes |4_2^-\rangle, |4_1^-\rangle \otimes |0_2\rangle\}, \quad (10)$$

$$\Gamma_4 = \{|1_1^+\rangle \otimes |3_2^-\rangle, |3_1^-\rangle \otimes |1_2^+\rangle\}, \quad (11)$$

$$\Gamma_5 = \{|2_1^+\rangle \otimes |2_2^-\rangle, |2_1^-\rangle \otimes |2_2^+\rangle\}, \quad (12)$$

$$\Gamma_6 = \{|3_1^+\rangle \otimes |1_2^-\rangle, |1_1^-\rangle \otimes |3_2^+\rangle\}, \quad (13)$$

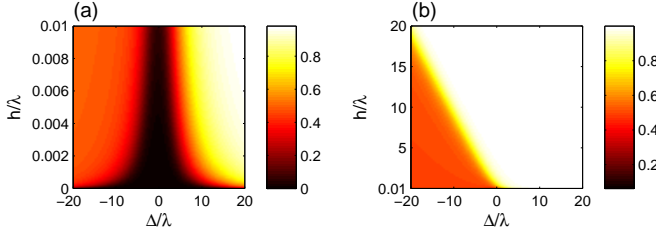


FIG. 2: (Color online) [(a)-(b)] ΔN_1 as a function of Δ and h for the lowest energy state of the system when $N = 4$.

$$\Gamma_7 = \{|4_1^+\rangle \otimes |0_2\rangle, |0_1\rangle \otimes |4_2^+\rangle\}, \quad (14)$$

$$\Gamma_8 = \{|2_1^+\rangle \otimes |2_2^+\rangle\}. \quad (15)$$

Figure 1 shows the energy differences between consecutive energy levels.

Note that the energetic ordering of the subspaces $\Gamma_1 \rightarrow \Gamma_8$ is independent of the parameters λ and Δ . Notably, the gap between the lowest two energy levels is $E_{\Delta 1,2} = \frac{1}{2}|\sqrt{\Delta^2 + 12\lambda^2} - 2\sqrt{\Delta^2 + 8\lambda^2} + \sqrt{\Delta^2 + 4\lambda^2}|$, which approaches zero in the limits of both large positive and large negative detuning. This is contrary to the situation with total excitation number $N = 2$, in which the energy difference between different subspaces goes to infinity in the limit of large negative detuning [18, 23].

In order to distinguish the insulator and superfluid states in the lowest energy state of the system, we use the variance of the total excitation number on the first site \hat{N}_1 as a measure [6]:

$$\Delta N_1 = \langle \hat{N}_1^2 \rangle - \langle \hat{N}_1 \rangle^2, \quad (16)$$

where $\hat{N}_1 = a_1^\dagger a_1 + |e_1\rangle\langle e_1|$. A plot of ΔN_1 as a function of the detuning Δ and the photon hopping strength h is given in Fig. 2.

For $\Delta = 0$ and $h/\lambda \ll 1$, the atom-field interaction on one site shifts the frequency of the field, causing a photon blockade effect [9, 10] that leads to a large energy gap between the lowest two subspaces (Γ_1 and Γ_2) and prevents additional photons from entering the site. For $N = 2$ [18] the gap between the lowest two energy levels is given by $(2 - \sqrt{2})\lambda \simeq 0.59\lambda$. For $N = 4$ the corresponding gap is $(2\sqrt{2} - 1 - \sqrt{3})\lambda \simeq 0.096\lambda$, indicating that the hopping strength h needed to overcome the photon blockade is much smaller as compared with the case with $N = 2$. For $N = 4$, the lowest energy state of the system is approximated by $|2_1^-\rangle \otimes |2_2^-\rangle$, as shown in Fig. 3(a), where N_A is defined as the total excitation number of both atoms. This lowest energy state contains two excitations on each site and the state $|2_j^-\rangle$ is a maximally entangled state of the atom and the field on the j th site. Fig. 3(b) shows that in the lowest energy state the components with both atoms in the ground state and both in the excited state are equally populated, which corresponds to a polaritonic insulator state for exact resonance and small hopping [18].

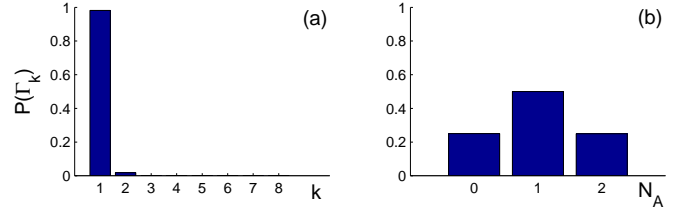


FIG. 3: (Color online) (a) Probability distribution $P(\Gamma_k)$ [$k = 1, 2, \dots, 8$, corresponding to Eqs. (8)-(15)] of the lowest energy state with respect to the eigenspace of the Hamiltonian without hopping. (b) Probability distribution of the total atomic excitation number N_A . Parameters for both subfigures are $\Delta = 0$ and $h = \lambda/200$.

When $h = 0$, the energy gap $E_{\Delta 1,2}$ goes to zero in the limit $\Delta/\lambda \rightarrow \pm\infty$, meaning the lowest level of the system is degenerate. However, a small but nonzero hopping value ($h/\lambda \ll 1$) breaks this degeneracy, leading to a unique state with the lowest energy. This lowest energy state involves a superposition of polaritonic states, as seen in Fig. 4(a) and (c). The eigenstates of each independent site vary with the sign of the detuning, as seen in Fig. 4(b) and (d). For $\Delta/\lambda \rightarrow \infty$, the lowest energy state is approximately $|g_1 g_2\rangle \otimes [\frac{\sqrt{10}}{5} |2_1 2_2\rangle + \frac{1}{2} (|1_1 3_2\rangle + |3_1 1_2\rangle) - \frac{\sqrt{5}}{10} (|4_1 0_2\rangle + |0_1 4_2\rangle)]$, which is a delocalized photon state (photonic superfluid state). For $\Delta/\lambda \rightarrow -\infty$, the lowest energy state approximates $|e_1 e_2\rangle \otimes [\frac{\sqrt{2}}{2} |1_1 1_2\rangle + \frac{1}{2} (|0_1 2_2\rangle + |2_1 0_2\rangle)]$, which is a coexisting state with the characteristics of both photonic superfluid and atomic insulator. This near-unity photon number is very different from the system with total excitation number $N = 2$ investigated in Ref. [18] in which the lowest energy state is an atomic insulator state for small hopping and large negative detuning. This result has a simple physical explanation. When Δ is negative, the energy of the atomic excitation is lower than that of the photon. To ensure the energy to be minimum, two excitations should first be occupied by the atoms. The remaining two excitations are populated in the photonic modes, with the distribution being determined by the competition of the non-linear Kerr effect induced by the dispersive atom-cavity interaction and photon hopping.

Therefore, when the hopping is small, the lowest energy state of the system undergoes quantum phase transitions from a polaritonic insulator state near exact resonance to a photonic superfluid state at large positive detuning.

In order to identify the polaritonic superfluid phase in the lowest energy state, we take the variance of the excitation number of the first atom ΔN_{1A} as a measure, where $\Delta \hat{N}_{1A} = |e_1\rangle\langle e_1|$; the results are plotted in Fig. 5(a) and (c). ΔN_{1A} is zero for the atomic insulator state, but is nonzero for a state with polaritonic characteristics. There are two regions with $\Delta N_{1A} = 0$ in Fig. 5(a): The region with $-\Delta \gg \lambda$ is a coexisting state where two excitations are occupied by the atoms and localized at

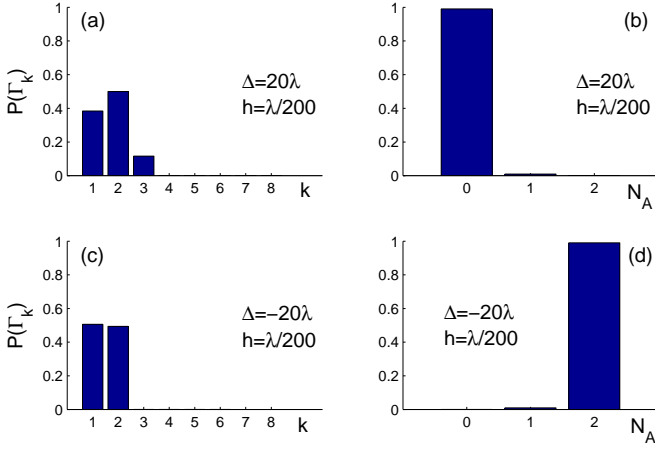


FIG. 4: (Color online) [(a)-(c)] Probability distribution $P(\Gamma_k)$ [$k = 1, 2, \dots, 8$, corresponding to Eqs. (8)-(15)] of the lowest energy state with respect to the eigenspace of the Hamiltonian without hopping. [(b)-(d)] Probability distribution of the total atomic excitation number N_A .

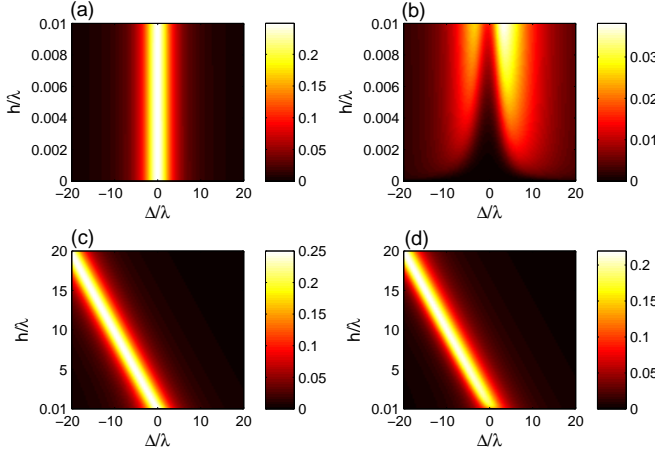


FIG. 5: (Color online) [(a)-(c)] ΔN_{1A} as a function of Δ and h for the lowest energy state of the system when $N = 4$. [(b)-(d)] Product $\Delta N_1 \Delta N_{1A}$ as a function of Δ and h for the lowest energy state of the system when $N = 4$.

different sites, while the region with $\Delta \gg \lambda$ is the photonic superfluid state where no atom is excited.

The product $\Delta N_1 \Delta N_{1A}$ may be used to characterize the polaritonic superfluid state as shown in Fig. 5(b), (c), and (d). $\Delta N_1 \Delta N_{1A}$ is zero for the polaritonic insulator state, but is nonzero for the polaritonic superfluid state. It is apparent that the polaritonic superfluid state appears in the near-resonance region for small hopping.

B. Small atom-field interaction

In this section, the atom-field interaction in H is taken as a perturbation in order to analyze the system in the large-hopping regime.

When $h < -\Delta$, the lowest eigenenergy is $4w_c + 2\Delta - 2h$,

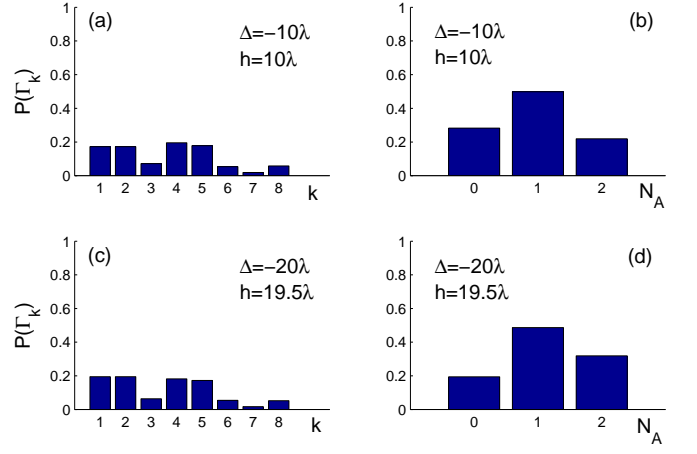


FIG. 6: (Color online) [(a)-(c)] Probability distribution $P(\Gamma_k)$ [$k = 1, 2, \dots, 8$, corresponding to Eqs. (8)-(15)] of the lowest energy state with respect to the eigenspace of the Hamiltonian without hopping. [(b)-(d)] Probability distribution of the total atomic excitation number N_A .

and the corresponding eigenstate is the coexisting state $|\varphi_{co}\rangle = |e_1 e_2\rangle \otimes [\frac{\sqrt{2}}{2} |1_1 1_2\rangle - \frac{1}{2} (|0_1 2_2\rangle + |2_1 0_2\rangle)]$. This coexisting state is similar to that in the regime $\Delta/\lambda \rightarrow -\infty$ for small hopping investigated in Sec. II A., but is very different from the case of $N = 2$ discussed in Ref. [18] in which the lowest energy state is still an atomic insulator state for large negative detuning. For $h > -\Delta$ the lowest eigenenergy is $4w_c - 4h$, and the corresponding eigenstate is the photonic superfluid state $|\varphi_{ps}\rangle = |g_1 g_2\rangle \otimes [\frac{1}{4} (|4_1 0_2\rangle + |0_1 4_2\rangle) - \frac{1}{2} (|1_1 3_2\rangle + |3_1 1_2\rangle) + \frac{\sqrt{6}}{4} |2_1 2_2\rangle]$.

When $h = -\Delta$ and $\lambda = 0$, the lowest level of energy $4w_c - 2h$ exhibits four-fold degeneracy with the eigenstates $|\varphi_{co}\rangle$, $|\varphi_{ps}\rangle$, $|\varphi_1\rangle$, and $|\varphi_2\rangle$, where

$$|\varphi_1\rangle = \frac{2\sqrt{2}}{3} |e_1 g_2\rangle \otimes [(|3_1 0_2\rangle - |0_1 3_2\rangle) - \sqrt{3}(|2_1 1_2\rangle - |1_1 2_2\rangle)], \quad (17)$$

$$|\varphi_2\rangle = \frac{2\sqrt{2}}{3} |g_1 e_2\rangle \otimes [(|3_1 0_2\rangle - |0_1 3_2\rangle) - \sqrt{3}(|2_1 1_2\rangle - |1_1 2_2\rangle)]. \quad (18)$$

The small atom-field interaction breaks this four-fold degeneracy, leading to a unique state with the lowest energy. As shown in Fig. 5(d), for large hopping this eigenstate can be identified as a polaritonic superfluid state when $h \simeq -\Delta$. The probability distributions of such two eigenstates with respect to the eigenspace Γ_k and atomic excitation number N_A are plotted in Fig. 6. In this case all the polariton subspaces are occupied and atoms are partially excited, indicating that the superfluid state has polaritonic characteristics.

IV. TOTAL EXCITATION NUMBER N

In this section, we examine the dependence of the variances $\Delta N_1/N$, ΔN_{1A} , and $(\Delta N_1/N)\Delta N_{1A}$ on the total excitation number N in the lowest energy state with fixed hopping strength. Here, the relative total excitation number variance $\Delta N_1/N$ is used to eliminate the effect of simply expanding the total excitation number and characterize the polaritonic superfluid.(see Sec. IV.B. for discussion).

A. Small hopping

In Fig. 7, we plot different excitation number variances as functions of the detuning in the lowest energy state of the N -excitation coupled atom-cavity system under the small-hopping situation, where $N = 4, 6, 8, \dots, 30$.

Figure 7(a) shows the effect of increasing N on the relative total excitation number variance $\Delta N_1/N$. As before, $\Delta N_1/N > 0$ indicates a delocalized or superfluid-like state, while $\Delta N_1/N \sim 0$ indicates a localized, insulatorlike state. For large positive detuning, $\Delta N_1/N > 0$ corresponds to the photonic superfluid state, while for large negative detuning, $\Delta N_1/N > 0$ corresponds to the coexisting state with characteristics of both photonic superfluid and atomic insulator. In the limit of very large $|\Delta|$, as N increases, the relative excitation number variance decreases, and the transition from the insulator to superfluid becomes slower. Figure 7(a) also shows that the region over which $\Delta N_1/N \sim 0$, indicating an insulator state, narrows as N increases. The photon hopping strength is set to be $h = 10^{-4}\lambda$ similar to that of Sec. III.A. For arbitrary even $N > 2$ the gap between the lowest two states for $h = 0$ becomes $(2\sqrt{N} - \sqrt{N-1} - \sqrt{N+1})\lambda$, which goes to zero as $N \rightarrow \infty$. Therefore as N increases the photon blockade, which leads to the polaritonic insulator state, weakens and hence can be easily overcome by the photon hopping. When $N \geq 20$, the system can not stay in the insulator state for $h = 10^{-4}\lambda$.

Figure 7(b) shows the atomic excitation number variance ΔN_{1A} as a function of Δ/λ for different N . As N increases, the maximum value of ΔN_{1A} remains unchanged, but the region with nonzero ΔN_{1A} is broadened. The maximum variance of the atomic excitation number depends on the number of atoms in the system, which is independent of the excitation number N . However, the on-resonance Rabi frequency scales as $\sqrt{N}\lambda$. Therefore the range of detunings over which the atom-field interaction is large enough to produce polaritonic behavior increases with N .

The combined effect of these features is shown in Fig. 7(c), where the product of the two variances as a function of Δ/λ is plotted. Here a nonzero value corresponds directly to a polaritonic superfluid state. It is clear that the polaritonic superfluid region extends over a wider range of Δ values as N increases. The dip at $\Delta = 0$ that indi-

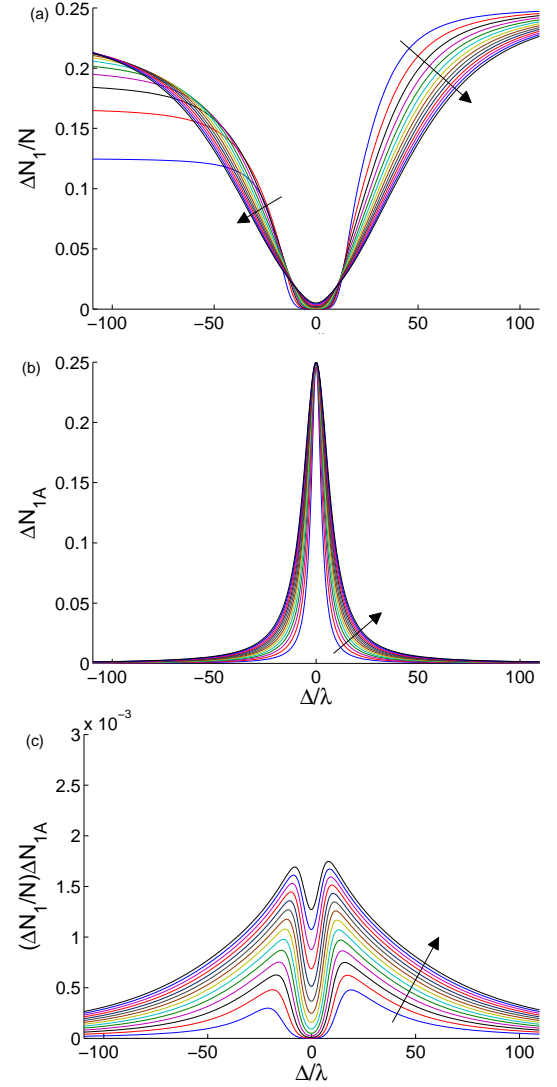


FIG. 7: (Color online) Excitation number variance as a function of the detuning Δ in the lowest energy state of the N -excitation ($N = 4, 6, 8, \dots, 30$) coupled atom-cavity system when $h = 10^{-4}\lambda$, where different excitation number variances are: (a) $\Delta N_1/N$; (b) ΔN_{1A} ; (c) $(\Delta N_1/N)\Delta N_{1A}$. The direction of the black arrows represents the increasing trend for N in these subfigures.

cates the transition to a polaritonic insulator state fails to go to 0 when $N \geq 20$, because, as discussed above, the value of h used in this plot is large enough to overcome the energy gap between the lowest two levels produced by the atom-cavity interaction when $N \geq 20$.

B. Small atom-field interaction

In Fig. 8, we plot various excitation number variances as functions of the detuning for the lowest energy state of the N -excitation coupled atom-cavity system, where $N = 4, 6, 8, \dots, 30$, in the small atom-field interaction regime.

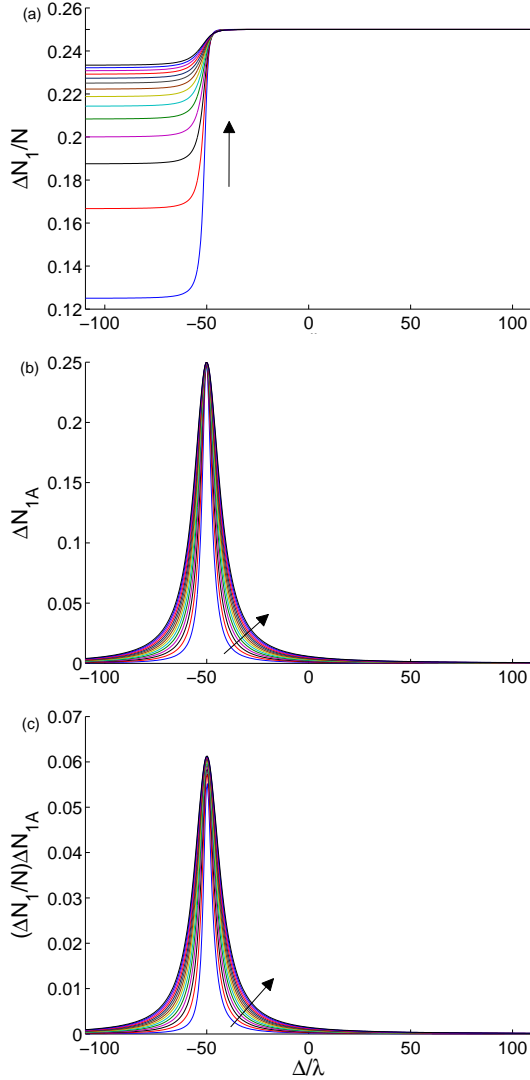


FIG. 8: (Color online) Excitation number variance as a function of the detuning Δ in the lowest energy state of the N -excitation ($N = 4, 6, 8, \dots, 30$) coupled atom-cavity system when $h = 50\lambda$, where different excitation number variances are: (a) $\Delta N_1/N$; (b) ΔN_{1A} ; (c) $(\Delta N_1/N)\Delta N_{1A}$. The direction of the black arrows represents the increasing trend for N in these subfigures.

Figure 8(a) shows that, in the region $\Delta \leq -h$, the relative excitation number variance $\Delta N_1/N$ becomes larger as N increases, indicating that the superfluid is enhanced. This is due to the fact that when Δ passes the critical point $\Delta_c = -h$, the values of $\Delta N_1/N$ all converge to the maximum $1/4$. Fig. 8(b) shows the atomic excitation number variance ΔN_{1A} as a function of Δ for different N when $h = 50\lambda$. The result is similar to that of Fig. 7(b), except that the nonzero region indicating that polaritonic states is now centered around $\Delta = -h$. The product of the two variances as a function of Δ/λ , plotted in Fig. 8(c), demonstrates the existence of a distinct polaritonic superfluid state in the vicinity of $\Delta = -h$. As N increases, the width of the polaritonic superfluid region

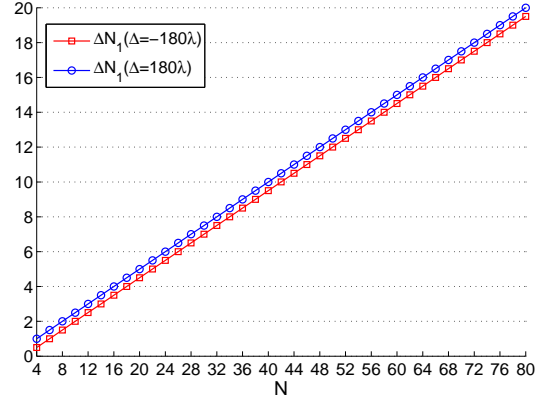


FIG. 9: (Color online) Excitation number variance ΔN_1 as a function of N in the lowest energy state for the large-detuning limits, where $h = 25\lambda$.

also increases. The maximum value of the variance product also increases with N . These features suggest that, in the large hopping limit, the polaritonic superfluid state may be easier to observe as the total excitation number increases.

In Fig. 9, we plot the variance of the total excitation number in the first site ΔN_1 as a function of the total excitation number for $\Delta \rightarrow -\infty$ and $\Delta \rightarrow \infty$. The results show that ΔN_1 has a linear dependence on N in both limits. We explain this result as follows. In the limit of large detuning, the two field modes effectively decouple from the atoms and can be approximately described by the interaction Hamiltonian:

$$H_{ph} = w_c(a_1^\dagger a_1 + a_2^\dagger a_2) + h(a_1^\dagger a_2 + a_2^\dagger a_1). \quad (19)$$

H_{ph} is easily diagonalized by defining the delocalized mode operators:

$$b_\pm = \frac{1}{\sqrt{2}}(a_1 \pm a_2), \quad (20)$$

where $[b_+, b_-] = 0$. In terms of these new operators, H_{ph} becomes:

$$H_{ph} = (w_c + h)b_+^\dagger b_+ + (w_c - h)b_-^\dagger b_-. \quad (21)$$

Assuming $h > 0$, the lowest energy state in its N -excitation subspace is given by:

$$|\psi_{g,N}^{ph}\rangle = \frac{1}{\sqrt{N!}}(b_-^\dagger)^N|0_1\rangle|0_2\rangle. \quad (22)$$

Thus, we straightforwardly calculate ΔN_1 in the large-detuning limits:

$$\Delta N_1 \simeq \frac{(N-2)}{4}, (\Delta \rightarrow -\infty) \quad (23)$$

$$\Delta N_1 \simeq \frac{N}{4}, (\Delta \rightarrow \infty) \quad (24)$$

which analytically demonstrates the linear relationship between ΔN_1 and N in the large-detuning limit.

For large positive Δ , the atoms are both in the ground states and so the lowest energy state of the system is given by $|\psi_{g,N}^{ph}\rangle$: all N excitations go into the delocalized mode b_- . On the other hand, for large negative Δ the atoms are both excited, leaving only $N - 2$ excitations in the field modes. In other words, in this regime the lowest energy state is the combination of the atomic insulator state and the photonic superfluid state. The effect of the atomic insulator state on the total excitation variance is weakened as N increases. This behavior is evident in Fig. 7(a) and Fig. 8(a). The state $|\psi_{g,N}^{ph}\rangle$ has a larger number variance for large N but is not in a sense any more delocalized, since the delocalized mode defined by b_- is maximally delocalized over the two sites. Larger N simply means there are more photons in the delocalized mode b_- , which explains the behavior shown in Fig. 8(a), where the curves for different N all converge to the same constant.

V. CONCLUSION

In summary, we have investigated the quantum phase transition behavior of polaritonic excitations in a multi-excitation coupled atom-cavity system. By examining our system with various parameters, we have identified different phases in the lowest energy state. The case of total excitation number $N = 4$ has been treated both

analytically and numerically, and the results are then generalized to the case with higher excitation numbers. In the small photon-hopping and the small atom-field interaction cases, we have identified an interesting co-existing phase involving characteristics of both photonic superfluid and atomic insulator. We find that the region where the system exhibits the polaritonic characteristic becomes broader as the total excitation number increases. Finally, we demonstrate that ΔN_1 has a linear dependence on N in the large-detuning limit. The results we have presented are not limited to the cavity QED system, and are general and applicable to all the systems, such as ion traps and circuit QED systems.

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